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Neutrinos in the Kerr and Robertson–Walker geometries

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Abstract. The perturbative behaviour of neutrinos is examined in the framework of the Hertz potential formalism in two important space–times, namely the Kerr and Robertson–Walker space–times. In particular, the angular functions of the neutrino field are studied in detail on the background of the Kerr geometry. It is found that the properties of the solutions depend crucially on the value of the separation constant $\bar{Q}$ defined in the text.

It is found that the Robertson–Walker model ($k = +1$) harbours a repulsive effective potential for neutrinos. The behaviour of the neutrinos in the cases of the spherically collapsing dust interior and the $k = -1$ Robertson–Walker model can be deduced from the $k = +1$ case. Also exact solutions in terms of known functions are obtained for the neutrino perturbations for the $k = 0$ case.

In the geometric optics limit ($\omega M \gg 1$) all the results agree with the classical ones obtained in the null geodesic formalism.

1. Introduction

The Hertz potential formalism for electromagnetic perturbations was given by Cohen and Kegeles (1974). This method was applied to perfect fluid space–times with local rotational symmetry (Dhurandhar *et al* 1980) and to the Gödel universe (Cohen *et al* 1980) for obtaining electromagnetic fields superposed on these background space–times. The approach consists in extracting full information about the fields from a single complex scalar which satisfies a decoupled differential equation. Subsequently, Kegeles and Cohen (1979) generalised the formalism to fields of arbitrary spin, in particular, the gravitational and neutrino perturbations. We use this formalism to study the neutrino fields in two important and astrophysically significant background metrics, namely, the Kerr and the Robertson–Walker. The more general case of the perfect fluid space–times with local rotational symmetry will be published elsewhere.

In § 2 we outline the salient features of the formalism necessary for our purpose and describe the general procedure for obtaining the neutrino field solutions in these space–times. Section 3 comprises a detailed investigation of the behaviour of neutrinos in the two geometries. First we treat the case of the Kerr metric and then the Friedmann solutions representing the cosmological models as well as the phenomenon of gravitational collapse. In § 4 we compare the results derived by our formalism with those of other formalisms. The results pertaining to gravitational collapse are compared with those of the usual Dirac formalism (Iyer *et al* 1982), which could with slight modification yield information on the cosmological metrics.

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2. The general Hertz potential formalism for neutrinos

The two-component Weyl neutrino equation is generalised to curved space–times and is reduced to a one-component equation for the Debye potential. With the addition of the appropriate gauge terms, a single decoupled equation is obtained for a complex scalar function ψ. The Weyl spinor describing the neutrino is then just given by a combination of the operations of differentiation and multiplication on this complex scalar. The computations are carried out with the use of the Newman–Penrose formalism. We merely state the relevant equation for ψ and mention the formulae which give the Weyl spinor in terms of ψ. The details of this procedure may be found in the paper by Kegeles and Cohen (1979).

With appropriate choice of the null tetrad (k^μ, n^μ, m^μ, $\bar{m}^\mu$), the equation for ψ in the usual notation is given by

$$[(\Delta+\mu-\gamma)(D+\bar{\varepsilon})-(\delta+\beta-\tau)(\bar{\delta}+\bar{\beta})]\psi=0, \tag{2.1}$$

where D, Δ, δ and $\bar{\delta}$ are the directional derivatives in the directions of k^μ, n^μ, m^μ and $\bar{m}^\mu$ respectively and μ, γ, ε, β and τ are the spin coefficients. The components of the Weyl spinor are given by

$$\phi_1=-(D+\bar{\varepsilon})\psi, \qquad \phi_2=-(\bar{\delta}+\bar{\beta})\psi. \tag{2.2}$$

The solution of equation (2.1) furnishes complete information on the behaviour of the neutrinos when coupled with equation (2.2).

3. Neutrino equations

3.1. *The Kerr space–time*

The Kerr space–time is described in geometrical units ($c=1$ and $G=1$) by the metric

$$\begin{aligned} ds^2=&-\left(1-\frac{2Mr}{|\Sigma|^2}\right)dt^2-\frac{4Mar\sin^2\theta}{|\Sigma|^2}\,dt\,d\varphi+\frac{|\Sigma|^2}{\Delta}dr^2+|\Sigma|^2\,d\theta^2\\ &+\sin^2\theta[r^2+a^2+(2Ma^2r\sin^2\theta)/|\Sigma|^2]\,d\varphi^2, \end{aligned} \tag{3.1}$$

where M is the mass and $a=J/M$ the angular momentum parameter, J is the angular momentum, $\Sigma=r+ia\cos\theta$ and $\Delta=r^2-2Mr+a^2$.

We make the following choice of the null tetrad which automatically aids in the separation of the variables when solving the equation for ψ. With the convention $v^\mu\equiv(v^t, v^r, v^\theta, v^\varphi)$, the tetrad consists of the following null vectors:

$$\begin{aligned} &k^\mu\equiv\Delta^{-1}(r^2+a^2, \Delta, 0, a), \qquad n^\mu\equiv\tfrac{1}{2}|\Sigma|^{-2}(r^2+a^2, -\Delta, 0, a),\\ &m^\mu\equiv(1/\sqrt{2}\Sigma)(ia\sin\theta, 0, 1, i\operatorname{cosec}\theta). \end{aligned} \tag{3.2}$$

The spin coefficients computed from equation (3.2) and necessary for our investigation are given by the relations

$$\begin{aligned} &\mu=-\frac{\Delta}{2|\Sigma|^2\bar{\Sigma}}, \qquad \gamma=\mu+\frac{r-M}{2|\Sigma|^2}, \qquad \varepsilon=0,\\ &\beta=\frac{1}{2\sqrt{2}}\frac{\cot\theta}{\Sigma}, \qquad \tau=-\frac{1}{\sqrt{2}}\frac{ia\sin\theta}{|\Sigma|^2}, \end{aligned} \tag{3.3}$$

where $\bar{\Sigma}$ denotes the complex conjugate of Σ. The choice of the above null tetrad along with the spin coefficients (3.3), when substituted into the equation (2.1), yields a decoupled equation for the complex scalar function ψ. Equation (2.1) assumes the form

$$\left[\frac{\Delta}{|\Sigma|^2}\left(\frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t}-\frac{\partial}{\partial r}+\frac{a}{\Delta}\frac{\partial}{\partial\varphi}-\frac{r-M}{\Delta}\right)\left(\frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}+\frac{a}{\Delta}\frac{\partial}{\partial\varphi}\right)\right.$$
$$-\frac{1}{\Sigma}\left(\mathrm{i}a\sin\theta\frac{\partial}{\partial t}+\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\cot\theta+\frac{\mathrm{i}a\sin\theta}{\bar{\Sigma}}\right)\frac{1}{\bar{\Sigma}}$$
$$\left.\times\left(-\mathrm{i}a\sin\theta\frac{\partial}{\partial t}+\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\cot\theta\right)\right]\psi=0. \tag{3.4}$$

From the Killing symmetries of the Kerr geometry the coordinates t and φ are ignorable, and accordingly we may set

$$\psi=\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}m\varphi}Z(r,\theta) \tag{3.5}$$

in (3.4) where ω and m are constants related to the energy and the azimuthal angular momentum respectively. After simplification (3.4) yields the equation

$$\left[\Delta\left(-\frac{\mathrm{i}\omega(r^2+a^2)}{\Delta}-\frac{\partial}{\partial r}+\mathrm{i}\frac{am}{\Delta}-\frac{r-M}{\Delta}\right)\left(-\frac{\mathrm{i}\omega(r^2+a^2)}{\Delta}+\frac{\partial}{\partial r}+\frac{\mathrm{i}am}{\Delta}\right)\right.$$
$$-\left(\frac{\partial}{\partial\theta}+a\omega\sin\theta-\frac{m}{\sin\theta}+\tfrac{1}{2}\cot\theta\right)$$
$$\left.\times\left(\frac{\partial}{\partial\theta}-a\omega\sin\theta+\frac{m}{\sin\theta}+\tfrac{1}{2}\cot\theta\right)\right]Z=0. \tag{3.6}$$

One finds that the r-dependent function Σ drops out from the 'angular part' of the differential operator in equation (3.4) after simplification. Therefore equation (3.6) can be further separated to give a pair of ordinary differential equations in the coordinates r and θ. If we set

$$Z(r,\theta)=R(r)S(\theta) \tag{3.7}$$

the separated equations for $R(r)$ and $S(\theta)$ are

$$\Delta\left(-\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\mathrm{i}\omega(r^2+a^2)}{\Delta}+\frac{\mathrm{i}am}{\Delta}-\frac{(r-M)}{\Delta}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\mathrm{i}\omega(r^2+a^2)}{\Delta}+\frac{\mathrm{i}am}{\Delta}\right)R+\alpha R=0, \tag{3.8}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\theta}+a\omega\sin\theta-\frac{m}{\sin\theta}+\tfrac{1}{2}\cot\theta\right)\left(\frac{\mathrm{d}}{\mathrm{d}\theta}-a\omega\sin\theta+\frac{m}{\sin\theta}+\tfrac{1}{2}\cot\theta\right)S+\alpha S=0. \tag{3.9}$$

Here α is the separation constant and is related to the one obtained by Carter (1968) in studying geodetic trajectories in the Kerr geometry. The equation (3.9) is the same as that of Chandrasekhar (1979). In the following discussion we study this equation in some detail.

3.1.1. The equation for $S(\theta)$. The exact solution for equation (3.7) seems difficult to obtain in terms of well known functions. However, for large values of ω, m and α it is possible to cast this equation in the effective potential form. For a solar mass Kerr black hole, large values of the separation constants ($\omega M \gg 1$ etc) are physically more viable. We neglect the terms $\frac{1}{2}\cot\theta$ appearing in equation (3.9) and also the linear terms in m and ω. We have the equation

$$\mathrm{d}^2S/\mathrm{d}\theta^2+(\alpha+2am\omega-a^2\omega^2\sin^2\theta-m^2/\sin^2\theta)S=0. \tag{3.10}$$

If one compares equation (3.10) with the first integral of the null geodesic equations for p_θ, where p_θ is the θ component of the particle's 4-momentum, one immediately sees that α plays a role similar to K (Carter 1968). If, in analogy with the classical considerations, we define Q by the equation

$$\alpha=(a\omega-m)^2+Q, \tag{3.11}$$

the equation for $S(\theta)$ becomes

$$\mathrm{d}^2S/\mathrm{d}\theta^2+[Q-\cos^2\theta(m^2/\sin^2\theta-a^2\omega^2)]S=0. \tag{3.12}$$

Setting $\bar{Q}=Q/\omega^2$ and $\bar{m}=m/\omega$, the equation can be written as

$$\mathrm{d}^2S/\mathrm{d}\theta^2+\omega^2[\bar{Q}-V(\theta)]S=0, \tag{3.13}$$

where

$$V(\theta)=\cos^2\theta(\bar{m}^2/\sin^2\theta-a^2). \tag{3.14}$$

We may note here that by putting $\bar{S}=(\sin\theta)^{1/2}S$ equation (3.9) can be written exactly as

$$\frac{\mathrm{d}^2\bar{S}}{\mathrm{d}\theta^2}+\left[\omega^2\bar{Q}-\omega^2\cos^2\theta\left(\frac{\bar{m}^2}{\sin^2\theta}-a^2\right)-\omega\cos\theta\left(\frac{\bar{m}}{\sin^2\theta}+a\right)\right]\bar{S}=0.$$

Then, with the approximations outlined above, this equation reduces to (3.13).

Oscillation of the wavefunction indicates that the neutrino is free to travel, while the damping represents the inability of the neutrino to enter the particular region. The solutions for equation (3.13) can be obtained in the WKB approximation, since ω is large. The solutions are oscillatory when $\bar{Q}>V(\theta)$; otherwise they are damped. Therefore, it is necessary to study the potential function $V(\theta)$ for various values of the parameters $\bar{m}$ and a.

The potential function $V(\theta)$ is symmetric about $\theta=\pi/2$. For $\bar{m}=0$, we must have $\bar{Q}>-a^2\cos^2\theta$ for solutions to be oscillatory. Two cases arise according as $|\bar{m}|\geqslant a$ or $|\bar{m}|<a$.

(i) $|\bar{m}|\geqslant a$

In this case $V(\theta)\geqslant 0$ with the minimum value zero at $\theta=\pi/2$. The necessary condition for solutions to be oscillatory is that $\bar{Q}$ be positive. For a given choice of $\bar{Q}=\bar{Q}_0>0$, $S(\theta)$ is oscillatory between $\theta=\theta_0$ and $\theta=\pi-\theta_0$ where θ_0 is the root of the equation $V(\theta)=Q_0$. This region of oscillation includes the equatorial plane $\theta=\pi/2$. As $\theta\to 0$ or π, $V(\theta)\to\infty$. Therefore, the angular function $S(\theta)$ is damped near the axis.

(ii) $|\bar{m}|<a$

The shape of $V(\theta)$ is more complicated. $V(\theta)$ has two minima at $\theta_1=\sin^{-1}(\bar{m}/a)^{1/2}$ and $\theta_2=\pi-\sin^{-1}(\bar{m}/a)^{1/2}$, where the inverse trigonometric sines have values which lie between 0 and $\pi/2$. $V(\theta_1)=V(\theta_2)=-(a-\bar{m})^2$ is the value of $V(\theta)$ at these minima. $V(\theta)$ vanishes at $\theta=\pi/2$ and grows without bound as θ approaches zero or π. The solutions with $-(a-\bar{m})^2<\bar{Q}<0$ are oscillatory in regions which lie on either side of the equatorial plane while the solutions with $\bar{Q}>0$ resemble those of case (i). We get a degenerate case as we let $\bar{m}\to 0$ in the above considerations.

The graphs of $V(\theta)$ for the two cases $|\bar{m}|\lessgtr a$ and the degenerate case $\bar{m}=0$ are shown in figure 1.

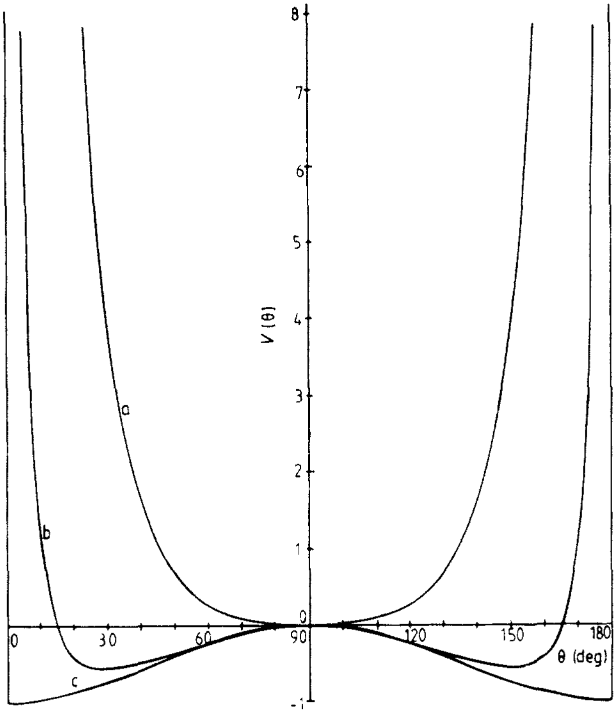

Figure 1 The effective potential curves V as a function of θ are drawn for different values of the parameter $\bar{m}/a$, namely (a) 1.25, (b) 0.25 and (c) 0. The curves exhibit essentially two types of behaviour according as $\bar{m}/a$ is less than or greater than unity. We have depicted one typical curve for each of the two cases. The degenerate case of $m=0$ is also shown. Curve (b) possesses two minima, one on each side of the equatorial plane $\theta=90°$, while curve (a) has only one minimum at $\theta=90°$. This reflects strongly on the behaviour of $S(\theta)$.

3.1.2. The radial equation. The radial equation in general has been discussed in fuller detail in the literature (Chandrasekhar 1979) than the angular equation in θ. We therefore present here a brief account of the relevant aspects of the radial equation.

Equation (3.8) may be expanded to give

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{1}{2}\frac{1}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}r}\frac{\mathrm{d}R}{\mathrm{d}r}+\bigg([\omega(r^2+a^2)-am]^2$$

$$+\frac{\mathrm{i}(r-M)}{\Delta^2}[\omega(r^2+a^2)-am]-\frac{2\mathrm{i}\omega r}{\Delta}-\frac{\alpha}{\Delta}\bigg)R=0. \qquad (3.15)$$

The term in the first derivative of R can be made to disappear by defining a new dependent variable $\bar{R}(r)$ by the relation

$$\bar{R}(r)=R(r)\Delta^{1/4}. \qquad (3.16)$$

The equation for $\bar{R}$ then reads

$$\mathrm{d}^2\bar{R}/\mathrm{d}r^2 + f(r)\bar{R} = 0, \tag{3.17}$$

where

$$f(r) = \Delta^{-2}\{[\omega(r^2+a^2) - am]^2 + \mathrm{i}(r-M)[\omega(r^2+a^2) - am] + \tfrac{3}{4}(r-M)^2 - \Delta(\alpha + \tfrac{1}{2} + 2\mathrm{i}\omega r)\}.$$

In the limit $\omega M \gg 1$ the equation resembles the first integral of the null geodesic equation in r with ω replaced by E, m by L_z and α by K. One simply considers terms quadratic in ω and m and the term in α and neglects all other terms. The resulting equation is

$$\mathrm{d}^2\bar{R}/\mathrm{d}r^2 + (\omega^2/\Delta^2)[(r^2+a^2-a\tilde{m})^2 - \alpha\Delta/\omega^2]\,\bar{R} = 0. \tag{3.18}$$

To obtain the solutions for $R(r)$ one must resort to numerical integration.

3.2. *The Robertson–Walker space–times*

The Robertson–Walker space–times are described by the geometry

$$\mathrm{d}s^2 = \mathrm{d}T^2 - S^2(T)[\mathrm{d}R^2/(1-kR^2) + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)], \tag{3.19}$$

where $k = \pm 1, 0$, and (R, θ, φ) are the comoving spatial coordinates of a particle; T is the cosmological time which is also the proper time for the particle and $S(T)$ is the expansion factor. The $k = +1$ model represents the closed universe while the $k = 0$ and $k = -1$ models represent open universes which expand for all times T. With a slight modification of the metric in (3.19) one obtains the model of a spherically symmetric object collapsing under its own gravitational pull.

We shall first treat the $k = +1$ and $k = 0$ cases in detail and indicate the calculations for $k = -1$ and the case of the collapsing object. Although it may be possible to examine all cases simultaneously by defining variables which possess different functional forms in each of the cases, it seems instructive to study one particular case for the sake of concreteness and merely notice how the solutions are modified in the rest of the models.

3.2.1. The $k = 1$ model. It is convenient to define two new variables u and η by the relations

$$R = \sin u, \qquad \eta = \int \frac{\mathrm{d}T}{S(T)}. \tag{3.20}$$

Then (3.19) assumes the form

$$\mathrm{d}s^2 = S^2(\eta)[\mathrm{d}\eta^2 - \mathrm{d}u^2 - \sin^2 u(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)]. \tag{3.21}$$

The metric given by (3.21) falls into the category of perfect fluid space–times with local rotational symmetry (Dhurandhar *et al* 1980). Following the notation of this reference, we have $F = 1/S$, $X = S$, $Y = S\sin u$, $t = \sin\theta$ and with the formulae for the spin coefficients and the directional derivatives given there we list the quantities necessary for equation (2.1):

$$\mu=\frac{1}{\sqrt{2}}\frac{1}{S}\left(\cot u-\frac{S'}{S}\right),\qquad \gamma=\frac{1}{2\sqrt{2}}\frac{S'}{S^2},\qquad \varepsilon=-\frac{1}{2\sqrt{2}}\frac{S'}{S^2},$$

$$\beta=\frac{1}{2\sqrt{2}}\frac{\cot\theta}{S\sin u},\qquad \tau=0,\qquad \sqrt{2}D=-\frac{1}{S}\frac{\partial}{\partial\eta}-\frac{1}{S}\frac{\partial}{\partial u},$$

$$\sqrt{2}\Delta=-\frac{1}{S}\frac{\partial}{\partial\eta}+\frac{1}{S}\frac{\partial}{\partial u},\qquad \sqrt{2}\delta=\frac{1}{S\sin u}\left(\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}\right),\tag{3.22}$$

where the prime denotes differentiation with respect to η.

The equation for the scalar ψ is

$$\left[\left(-\frac{1}{S}\frac{\partial}{\partial\eta}+\frac{1}{S}\frac{\partial}{\partial u}+\frac{\cot u}{S}-\frac{3}{2}\frac{S'}{S^2}\right)\left(-\frac{1}{S}\frac{\partial}{\partial\eta}-\frac{1}{S}\frac{\partial}{\partial u}-\frac{1}{2}\frac{S'}{S^2}\right)-\frac{1}{S^2\sin^2 u}\right.$$
$$\left.\times\left(\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\cot\theta\right)\left(\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\cot\theta\right)\right]\psi=0.\tag{3.23}$$

The equation (3.23) can be easily separated out by setting

$$\psi=Z(\eta,u)\Theta(\theta,\varphi).\tag{3.24}$$

We have the following equations for Z and Θ:

$$\left[\frac{\partial^2}{\partial\eta^2}+\left(\frac{S'}{S}-\cot u\right)\frac{\partial}{\partial\eta}-\frac{\partial^2}{\partial u^2}-\cot u\frac{\partial}{\partial u}-\frac{1}{2}\frac{S'}{S}\cot u+\frac{1}{2}\frac{S''}{S}-\frac{1}{4}\left(\frac{S'}{S}\right)^2+\frac{c}{\sin^2 u}\right]Z=0,\tag{3.25}$$

$$\left(\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\cot\theta\right)\left(\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\cot\theta\right)\Theta+c\Theta=0,\tag{3.26}$$

where c is the separation constant.

The solutions for (3.26) can be given in terms of the Jacobi polynomials. We simply state the solutions,

$$\Theta=(1-\cos\theta)^{\alpha'/2}(1+\cos\theta)^{\beta'/2}P_n^{(\alpha',\beta')}(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\varphi},\tag{3.27}$$

where m is an integer and n a positive integer,

$$\alpha'=|m+\tfrac{1}{2}|,\qquad \beta'=|m-\tfrac{1}{2}|,$$

and the separation constant c takes the values given by

$$c=(n+m+\tfrac{1}{2})^2.\tag{3.28}$$

Equation (3.25) is easily separated if we change the dependent variable to $\bar{Z}$ by the transformation

$$\bar{Z}=\sqrt{S}Z.\tag{3.29}$$

Then (3.25) becomes

$$\left(\frac{\partial^2}{\partial\eta^2}-\cot u\frac{\partial}{\partial\eta}-\frac{\partial^2}{\partial u^2}-\cot u\frac{\partial}{\partial u}+\frac{c}{\sin^2 u}\right)\bar{Z}=0.\tag{3.30}$$

Since η is ignorable in equation (3.30), after setting $\bar{Z} = \mathrm{e}^{\mathrm{i}\omega\eta} Z_\omega(u)$, we obtain an ordinary differential equation for Z_ω:

$$\frac{\mathrm{d}^2 Z_\omega}{\mathrm{d}u^2} + \cot u \frac{\mathrm{d}Z_\omega}{\mathrm{d}u} + \left(\omega^2 + \mathrm{i}\omega \cot u - \frac{c}{\sin^2 u}\right) Z_\omega = 0. \tag{3.31}$$

With the further transformation

$$\hat{Z}_\omega(u) = (\sin u)^{1/2} Z_\omega(u) \tag{3.32}$$

we obtain an equation for $\hat{Z}_\omega$ in a 'Schrödinger' form

$$\mathrm{d}^2\hat{Z}_\omega/\mathrm{d}u^2 + (\omega^2 - c/\sin^2 u + \tfrac{1}{4}\cot^2 u + \tfrac{1}{2} + \mathrm{i}\omega \cot u)\hat{Z}_\omega = 0. \tag{3.33}$$

The solution to (3.33) can be obtained by the WKB approximation when it is valid. For large values of ω and c (in the geometric optics approximation) the last three terms in the parentheses may be neglected and the equation becomes

$$\mathrm{d}^2\hat{Z}_\omega/\mathrm{d}u^2 + (\omega^2 - c/R^2)\hat{Z}_\omega = 0. \tag{3.34}$$

The solutions to (3.34) compare well with the classical null trajectories.

We now indicate how the above computations may be applied to the geodetic gravitational collapse of a spherical object. The Friedmann interior metric for dust (pressure $p = 0$) is given by

$$\mathrm{d}s^2 = \mathrm{d}T^2 - S^2[\mathrm{d}R^2/(1-\alpha R^2) + R^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2)] \tag{3.35}$$

where

$$\alpha = 2m/R_\mathrm{b}^3$$

and R_b is the maximum radius of the object in the exterior Schwarzschild coordinates at the beginning of the collapse. We replace S by $S/\sqrt{\alpha}$ in the above calculations and define u and η by the relations

$$\sin u = \sqrt{\alpha} R, \qquad \eta = \int \frac{\sqrt{\alpha}\, \mathrm{d}T}{S}. \tag{3.36}$$

Then

$$T = \alpha^{-1/2}(\tfrac{1}{2}\eta - \sin\tfrac{1}{2}\eta \cos\tfrac{1}{2}\eta), \qquad S = \sin^2(\tfrac{1}{2}\eta), \tag{3.37}$$

where η varies from π to 2π as the collapse progresses.

The results are essentially the same with the effective potential for the radial function varying as $1/R^2$ just as in equation (3.34).

Finally the neutrino behaviour for the $k = -1$ case can be easily obtained from the equations corresponding to $k = +1$. A formal replacement of R by $\mathrm{i}R$ and S by $\mathrm{i}S$ and similar transformations for the intermediate variables η and u yield equations analogous to the $k = +1$ case. However, the physical significance is different and can be conveniently obtained from the resulting equations.

3.2.2. The $k = 0$ model. We shall not enter into full discussion of the equations as was done for $k = +1$, but mention the salient steps which lead to the final solution.

The geometry is described by the line element

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + S^2[\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2)]. \tag{3.38}$$

As before we define a new time coordinate η by

$$\eta=\int\frac{\mathrm{d}T}{S}. \tag{3.39}$$

We will not mention here the spin coefficients or the directional derivatives, but directly write down the equation satisfied by the Hertz potential ψ. The equation (2.1) becomes

$$\left[\left(-\frac{1}{S}\frac{\partial}{\partial\eta}+\frac{1}{S}\frac{\partial}{\partial R}+\frac{1}{RS}-\frac{3}{2}\frac{S'}{S^2}\right)\left(-\frac{1}{S}\frac{\partial}{\partial\eta}-\frac{1}{S}\frac{\partial}{\partial R}-\frac{1}{2}\frac{S'}{S^2}\right)-\frac{1}{R^2S^2}\right.$$
$$\left.\times\left(\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\cot\theta\right)\left(\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\cot\theta\right)\right]\psi=0. \tag{3.40}$$

The separation of the (R, η) coordinates and the angular coordinates (θ, φ) is achieved by setting

$$\psi=Z(\eta, R)\Theta(\theta, \varphi). \tag{3.41}$$

The two separated equations for Z and Θ are given by

$$\left(-\frac{1}{S}\frac{\partial}{\partial\eta}+\frac{1}{S}\frac{\partial}{\partial R}+\frac{1}{RS}-\frac{3}{2}\frac{S'}{S^2}\right)\left(-\frac{1}{S}\frac{\partial}{\partial\eta}-\frac{1}{S}\frac{\partial}{\partial R}-\frac{1}{2}\frac{S'}{S^2}\right)Z+c\frac{Z}{R^2S^2}=0, \tag{3.42}$$

$$\left(\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\cot\theta\right)\left(\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\cot\theta\right)\Theta+c\Theta=0. \tag{3.43}$$

Here c is the separation constant. Equation (3.43) is the same as equation (3.26) for the case $k=+1$ and therefore we shall not discuss it further. The radial-temporal equation is however different. With the choice of a new dependent variable $\bar{Z}$ defined by

$$\bar{Z}=\sqrt{S}Z \tag{3.44}$$

we obtain an equation in which η is an ignorable coordinate. The equation for $\bar{Z}$ is

$$\left(\frac{\partial^2}{\partial\eta^2}-\frac{1}{R}\frac{\partial}{\partial\eta}-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}+\frac{c}{R^2}\right)\bar{Z}=0. \tag{3.45}$$

As in the earlier case, we define $\hat{Z}_\omega$ through the relation

$$\bar{Z}=(\mathrm{e}^{\mathrm{i}\omega\eta}/\sqrt{R})\hat{Z}_\omega. \tag{3.46}$$

Then equation (3.45) transforms to

$$\mathrm{d}^2\hat{Z}_\omega/\mathrm{d}R^2+[\omega^2+\mathrm{i}\omega/R-(c-\tfrac{1}{4})/R^2]\hat{Z}_\omega=0. \tag{3.47}$$

The solutions to this equation can be written in terms of confluent hypergeometric functions,

$$\hat{Z}_\omega\sim\mathrm{e}^{\mathrm{i}\omega R}(\omega R)^{1/2+\sqrt{c}}\begin{cases}F(\sqrt{c}, 1+2\sqrt{c}, 2\mathrm{i}\omega R)\\ U(\sqrt{c}, 1+2\sqrt{c}, 2\mathrm{i}\omega R).\end{cases} \tag{3.48}$$

4. Discussion

As can be seen from the foregoing calculations, the Hertz potential formalism provides a convenient framework for exploring the behaviour of neutrinos in the Kerr and Robertson–Walker space–times. The equation of the angular function $S(\theta, \varphi)$ in the Kerr geometry is the same as that obtained by Chandrasekhar (1979). We have studied this equation in detail since to the best of our knowledge it has not appeared in the literature. It is found that the pattern of oscillation and damping of the solutions depends critically on the value of the separation constant Q. The solutions are always damped near the axis irrespective of the values of Q, while there are regions of oscillation on either side of the equatorial plane if Q is negative. The radial equation has been investigated sufficiently in the literature. We only remark here that in the high-frequency approximation the results agree with those obtained by classical means.

In the Robertson–Walker models we have examined the $k=0$ and $k=+1$ cases fully, while we have indicated the calculations and results for $k=-1$ and for the case of a collapsing dust interior with spherical symmetry. The behaviour compares favourably with that obtained either by the conventional Dirac formalism or by a purely classical treatment involving the usual equations.

In the geometric optics approximation the behaviour of the neutrino is similar to that of the electromagnetic field. However, if one does not resort to this approximation there is a difference in their propagation. This comes about because the neutrino and the photon possess different spins.

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